The following was taken from "How to Read and Do Proofs: An Introduction to Mathematical Thought Processes" by Daniel Snow, John Wiley & Sons, Inc. New York 2002

IB MATH HL SUMMER WORK:
Read through the notes and examples.
Complete the examples on the last page.
Due first day of class.

11.2 INDUCTION

In Chapter 5, you learned to use the choose method when the quantifier "for all" appears in the statement B. There is one special form of B containing the quantifier "for all" for which a separate technique known as **induction** is likely to be more successful.

How to Use Induction

You should consider induction seriously (even before the choose method) when B has the form:

For every integer $n \geq 1$, "something happens,"

where the something that happens is some statement, $P(n)$, that depends on the integer n. The following is an example:

$$\text{For all integers } n \geq 1,\ \underbrace{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}_{P(n)}, \quad \text{where } \sum_{k=1}^{n} k = 1 + \cdots + n.$$

When considering induction, the key words to look for are "integer" and "≥ 1."

One way to attempt proving such statements is to make an infinite list of statements, one for each of the integers starting from $n = 1$, and then prove each statement separately. While the first few statements on the list are usually easy to verify, the issue is how to check statement number n and beyond. For the foregoing example, the list is:

$$\begin{array}{llll}
P(1): & \sum_{k=1}^{1} k = \frac{1(1+1)}{2} & \text{or} & 1 = 1 \\
P(2): & \sum_{k=1}^{2} k = \frac{2(2+1)}{2} & \text{or} & 1 + 2 = 3 \\
P(3): & \sum_{k=1}^{3} k = \frac{3(3+1)}{2} & \text{or} & 1 + 2 + 3 = 6 \\
 & \vdots & & \\
P(n): & \sum_{k=1}^{n} k = \frac{n(n+1)}{2} & & \\
P(n+1): & \sum_{k=1}^{n+1} k = \frac{(n+1)[(n+1)+1]}{2} & = & \frac{(n+1)(n+2)}{2} \\
 & \vdots & &
\end{array}$$

Induction is a clever method for proving that each of these statements in the infinite list is true. Think of induction as a proof machine that starts with $P(1)$ and progresses down the list, proving each statement as it proceeds. Here is how the machine works.

The machine is started by verifying that $P(1)$ is true, which is easy to do for the foregoing example. Then $P(1)$ is fed into the machine. The machine uses the fact that $P(1)$ is true and automatically proves that $P(2)$ is true. You then put $P(2)$ into the machine. The machine uses the fact that $P(2)$ is true to reach the conclusion that $P(3)$ is true, and so on (see Figure 11.1).

Observe that, by the time the machine is going to prove that $P(n+1)$ is true, it will already have shown that $P(n)$ is true (from the previous step). Thus, in designing the machine, you can assume that $P(n)$ is true; your job is to make sure that $P(n+1)$ is also true. In summary, the following steps constitute a proof by induction.

The Steps of Induction

Step 1. Verify that $P(1)$ is true.
Step 2. Use the assumption that $P(n)$ is true to prove that $P(n+1)$ is true.

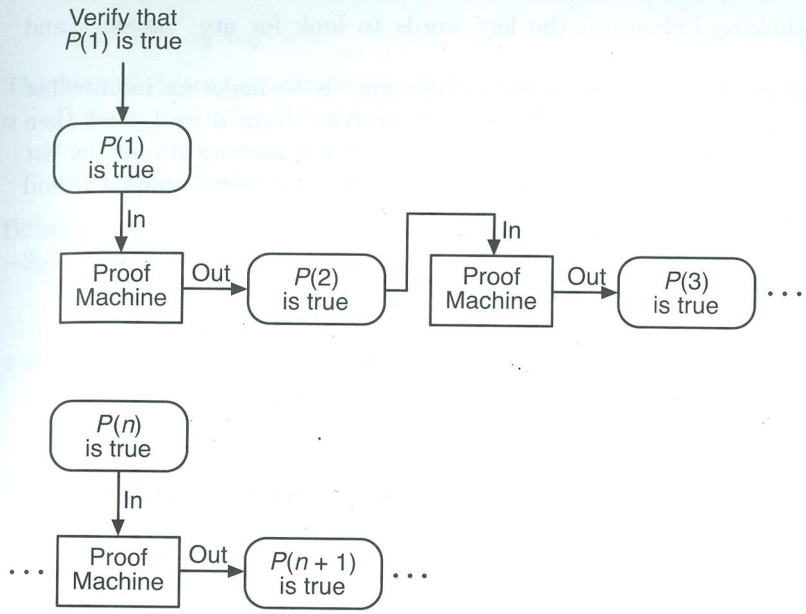

Fig. 11.1 The proof machine for induction.

To perform Step 1, replace n everywhere in $P(n)$ by 1. To verify that the resulting statement is true usually requires only some minor rewriting.

Step 2 is more challenging. You must reach the conclusion that $P(n+1)$ is true by using the assumption that $P(n)$ is true. There is a standard way of doing this. Begin by writing the statement $P(n+1)$, which you want to conclude is true. Because you are assuming that $P(n)$ is true, you should somehow try to rewrite the statement $P(n+1)$ in terms of $P(n)$, for then you can make use of the assumption that $P(n)$ is true. Using the assumption that $P(n)$ is true is referred to as **using the induction hypothesis**. On establishing that $P(n+1)$ is true, the proof is complete. The steps of induction are illustrated with the following proposition.

Proposition 19 *For every integer* $n \geq 1$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Analysis of Proof. When using the method of induction, it is helpful to write the statement $P(n)$, in this case:

$$P(n)\text{:}\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The first step in a proof by induction is to verify $P(1)$. Replacing n everywhere by 1 in $P(n)$, you obtain

$$P(1)\text{:}\ \sum_{k=1}^{1} k = \frac{1(1+1)}{2}.$$

With a small amount of rewriting, it is easy to verify this because

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}.$$

This step is often so easy that it is virtually omitted in the condensed proof simply by saying, "The statement is clearly true for $n = 1$."

The second step is more involved. You must use the assumption that $P(n)$ is true to reach the conclusion that $P(n+1)$ is true. The best way to proceed is to write the statement $P(n+1)$ by replacing carefully n everywhere in $P(n)$ with $n+1$ and rewriting a bit, if necessary. In this case

$$P(n+1): \sum_{k=1}^{n+1} k = \frac{(n+1)[(n+1)+1]}{2} = \frac{(n+1)(n+2)}{2}.$$

To reach the conclusion that $P(n+1)$ is true, begin with the left side of the equality in $P(n+1)$ and try to make that side look like the right side. In so doing, you should use the information in $P(n)$ by relating the left side of the equality in $P(n+1)$ to the left side of the equality in $P(n)$. Then you will be able to use the right side of the equality in $P(n)$. In this example,

$$P(n+1): \sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1).$$

Now you can use the assumption that $P(n)$ is true by replacing $\sum_{k=1}^{n} k$ with $n(n+1)/2$, obtaining

$$P(n+1): \sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

All that remains is a bit of algebra to rewrite $\frac{n(n+1)}{2} + (n+1)$ as $\frac{(n+1)(n+2)}{2}$, thus obtaining the right side of the equality in $P(n+1)$. The algebraic steps are:

$$\frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}.$$

Your ability to relate $P(n+1)$ to $P(n)$ so as to use the induction hypothesis that $P(n)$ is true determines the success of a proof by induction. If you are unable to relate $P(n+1)$ to $P(n)$, then you might wish to consider a different proof technique. On the other hand, if you can relate $P(n+1)$ to $P(n)$, you will find that induction is easier to use than almost any other technique. To illustrate this fact, you are asked in the exercises to prove Proposition 19 without using induction. Compare your proof with the condensed proof that follows.

Proof of Proposition 19. The statement is clearly true for $n = 1$. Assume the statement is true for n, that is, that $\sum_{k=1}^{n} k = n(n+1)/2$. Then

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \left(\sum_{k=1}^{n} k\right) + (n+1)\\ &= \tfrac{n(n+1)}{2} + (n+1)\\ &= (n+1)\left(\tfrac{n}{2}+1\right)\\ &= \tfrac{(n+1)(n+2)}{2}.\end{aligned}$$

Thus the statement is true for $n+1$ and so the proof is complete. □

Note that induction does not help you to discover the correct form of the statement $P(n)$. Rather, induction only verifies that a given statement $P(n)$ is true for all integers n greater than or equal to some initial one.

Some Variations on Induction

From the foregoing discussion, you know that in the second step in a proof by induction, you use the assumption that $P(n)$ is true to show that $P(n+1)$ is true. From a notational point of view, some authors prefer to use the assumption that $P(n-1)$ is true to show that $P(n)$ is true. These two approaches are identical—either can be used, depending on your notational preference. What is important is that you establish that if a general statement on the infinite list is true, then the next statement is also true.

When using induction, the first value for n need not be 1. For instance, you can use induction to prove that "for all integers $n \geq 5$, $2^n > n^2$." The only modification is that, to start the proof, you must verify $P(n)$ for the first given value of n. In this case, that first value is $n = 5$, so you have to check that $2^5 > 5^2$ (which is true because $2^5 = 32$ while $5^2 = 25$). The second step of the induction proof remains the same—you still have to show that if $P(n)$ is true (that is, $2^n > n^2$), then $P(n+1)$ is also true [that is, $2^{n+1} > (n+1)^2$]. In so doing, you can also use the fact that $n \geq 5$, if necessary.

Another modification to the basic induction method arises when you are having difficulty relating $P(n+1)$ to $P(n)$. Suppose, however, that you can relate $P(n+1)$ to $P(j)$, where $j < n$. In this case, you would like to use the fact that $P(j)$ is true but, can you assume that $P(j)$ is, in fact, true? The answer is yes. To see why, recall the analogy of the proof machine (look again at Figure 11.1) and observe that, by the time the machine has to show that $P(n+1)$ is true, the machine has already proved that all of the statements $P(1), \ldots, P(j), \ldots, P(n)$ are true. Thus, when trying to show that $P(n+1)$ is true, you can assume that $P(n)$ and all preceding statements are true. Such a proof is referred to as **generalized induction** and is illustrated now.

11.3 READING A PROOF

The process of reading and understanding a proof is demonstrated with the following proposition.

Proposition 20 *Any integer $n \geq 2$ can be expressed as a finite product of primes (see Definition 2 on page 24).*

Proof of Proposition 20. (For reference purposes, each sentence of the proof is written on a separate line.)

S1: The statement is clearly true for $n = 2$.
S2: Now assume the statement is true for all integers between 2 and n, that is, that any integer j with $2 \leq j \leq n$ can be expressed as a finite product of primes.
S3: If $n+1$ is prime, the statement is true for $n+1$.
S4: Otherwise, $n+1$ has a prime divisor, that is, there are integers p and q with p prime and $2 \leq q \leq n$ such that $n+1 = pq$.
S5: But by the induction hypothesis, q can be expressed as a finite product of primes and, therefore, so can $n+1$.

The proof is now complete. □

Analysis of Proof. An interpretation of statements $S1$ through $S5$ follows.

Interpretation of S1: *The statement is clearly true for $n = 2$.*

The author is performing the first step of induction by mentioning that the statement is true for the first value of n, namely, $n = 2$. The statement is true because 2 is itself prime.

Interpretation of S2: *Now assume the statement is true for all integers between 2 and n, that is, that any integer j with $2 \leq j \leq n$ can be expressed as a finite product of primes.*

The author is performing the second step of generalized induction by assuming that the statement is true for all integers between 2 and n. It remains to show that the statement is true for $n+1$.

Interpretation of S3: *If $n+1$ is prime, the statement is true for $n+1$.*

The author notes that the statement is true for $n+1$ when $n+1$ is prime, which is clearly correct. Presumably, the author will also show that the statement is true when $n+1$ is not prime.

Interpretation of S4: *Otherwise, $n+1$ has a prime divisor, that is, there are integers p and q with p prime and $2 \leq q \leq n$ such that $n+1 = pq$.*

The author is showing that the statement is true when $n+1$ is not prime. Specifically, the author is using the fact that when $n+1$ is not prime, $n+1$

has a prime divisor, p. The author then works forward from the fact that p divides $n+1$ using Definition 1 on 24 to claim that there is an integer q with $2 \leq q \leq n$ such that $n+1 = pq$.

Interpretation of S5: *But by the induction hypothesis, q can be expressed as a finite product of primes and, therefore, so can $n+1$.*

The author is applying the induction hypothesis to q, which is valid because q is an integer between 2 and n (see $S4$). Doing so yields that q is a finite product of primes. The author then notes that, as a result, $n+1 = pq$ is also a finite product of the prime p and the product of primes that constitute q. The generalized induction proof is now complete because the author has correctly established that the statement is true for $n+1$.

Summary

In this chapter, you have learned two special quantifier techniques: the uniqueness methods and induction.

Induction

Use induction when the statement you are trying to prove has the form, "For every integer $n \geq n_0$, $P(n)$," where $P(n)$ is some statement that depends on n. To apply the method of induction,

1. Verify that the statement $P(n)$ is true for n_0. (To do this, replace n everywhere in $P(n)$ by n_0, rewrite the resulting statement, and try to establish that $P(n_0)$ is true.)

2. Assume that $P(n)$ is true.

3. Write the statement $P(n+1)$ by replacing n everywhere in $P(n)$ with $n+1$. (Some rewriting may be necessary to express $P(n+1)$ in a clean form.)

4. Reach the conclusion that $P(n+1)$ is true. To do so, relate $P(n+1)$ to $P(n)$ and then use the fact that $P(n)$ is true. The key to using induction rests in your ability to relate $P(n+1)$ to $P(n)$.

Prove that $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{Z}^+$

P_n is "$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$"

If $n=1$ LHS $= 1^2 = 1$

RHS $= \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$

$\therefore P_1$ is true

Assume $P(k)$ is true

Want to show $1^2+2^2+3^2+\dots+k^2+(k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$\left\{ = \frac{(k+1)(k+2)(2k+3)}{6} \right\}$

$1^2+2^2+3^2+\dots+k^2+(k+1)^2$ substitute $P(k)$

$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$ GCF $k+1$

$\frac{(k+1)(k(2k+1)+6(k+1))}{6}$

$\frac{(k+1)(2k^2+k+6k+6)}{6}$

$\frac{(k+1)(2k^2+7k+6)}{6}$

$2k^2+4k+3k+6$

$2k(k+2)+3(k+2)$

$$\frac{(k+1)(2k+3)(k+2)}{6} \quad \checkmark$$

Thus $P(k+1)$ is true whenever $P(k)$ is true

and $P(1)$ is true $\therefore$ by PMI P_n is true

Prove that

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4} \text{ for all } n\in\mathbb{Z}^+$$

Let P(n) be

If $n=1$, LHS $\frac{1}{2\times 5}=\frac{1}{10}$

RHS $\frac{1}{6(1)+4}=\frac{1}{10}$ $\therefore$ P(1) is true

Assume P(K) is true

Want to show P(K+1)

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3(k+1)-1)(3(k+1)+2)}=\frac{k+1}{6(k+1)+4}$$

$$\left\{=\frac{k+1}{6k+10}\right\}$$

$$\underbrace{\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots\frac{1}{(3k-1)(3k+2)}}_{\text{substitute } P(K)}+\frac{1}{(3(k+1)-1)(3(k+1)+2)}$$

$$\frac{K}{6K+4}+\frac{1}{(3K+2)(3K+5)}$$ get LCD

$$\frac{K}{2(3K+2)}+\frac{1}{(3K+2)(3K+5)}$$

$$\frac{k(3k+5)}{2(3k+2)(3k+5)} + \frac{2}{2(3k+2)(3k+5)}$$

$$\frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$\frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}$$

$$\frac{k+1}{2(3k+5)}$$

$$\frac{k+1}{6k+10} \quad \checkmark$$

Thus P(k+1) is true whenever P(k) is true and P(1) is true. Therefore by PMI P(n) is true.

Prove that 4^n+2 is

divisible by 3 for $n \in \mathbb{Z}, n \geq 0$.

$P(n)$ is 4^n+2 is divisible by 3 for all $n \in \mathbb{Z}, n \geq 0$

(1st term)

$P(0)$: $4^0+2 = 3$ which is divisible by 3

So $P(0)$ is true

Assume $P(k)$ is true [ie. $4^n+2 = 3A$ where A is an integer

Want to show $P(k+1)$ is divisible by 3

$4^{k+1}+2$

$4(4^k)+2$ since $4^n+2 = 3A \Rightarrow 4^n = 3A-2$

$4(3A-2)+2$

$12A-8+2$

$12A-6$

$3(4A-2)$ $4A-2$ would be some integer

Thus $4^{k+1}+2$ is divisible by 3 if 4^{k+1} is divisible by 3. Hence

$P(k+1)$ is true whenever $P(k)$ is true and $P(0)$ is true

so by PMI $P(n)$ is true.

DUE FIRST DAY OF CLASS!

Prove the following:

1) $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

2) $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

3) $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots n \times n! = (n+1)! - 1$

4) $n^3 + 2n$ is divisible by 3 for all positive integers n.

5) $6^n - 1$ is divisible by 5 for all integers $n \geq 0$

6) $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$